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RAY METHOD OF SOLVING DYNAMIC PROBLEMS IN ELASTIC-VISCOPLASTIC MEDIA

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The velocity and stress fields behind the shock wave front are investigated by a ray method in a three-dimensional elastic viscoplastic medium with hardening. Recursion equations are obtained to determine the terms of the ray series. In the case of load propagation by a spherical shock wave an analytical solution is obtained for the velocity to second order accuracy in the distance along the normal from the wave front. The solution of wave problems in elasticity theory by a wave method has been elucidated in [1], and in a one-dimensional viscoelastic medium in [2].

1. An elastic viscoplastic body with a yield point dependent on the mean pressure, plastic strains, and plastic strain rate, is the model of the medium. The dependence of the yield point on the mean pressure and plastic strain rate is assumed linear, and on the plastic strain, arbitrary. The flow surface is taken as

$$f_1 = (s_{ij} - \eta_1 \varepsilon_{ij}^{*p})(s_{ij} - \eta_1 \varepsilon_{ij}^{*p}) - 2(k_0 - \alpha\sigma + \beta(e^p) + \eta_2(e^p)\,|\,\varepsilon^p\,|)^2 = 0 \qquad (1.1)$$

$$f_2 = -\sigma - \sigma_0 - \mu_1\,|\,e^p\,| - \eta_2\,|\,\varepsilon^p\,| = 0$$

Here

$$\sigma = {}^1/_3\,\sigma_{kk}, \qquad e^p = {}^1/_3\,e_{kk}^p, \qquad \varepsilon^p = {}^1/_3\,\varepsilon_{kk}^p$$

$$s_{ij} = \sigma_{ij} - \sigma\delta_{ij}, \qquad \varepsilon_{ij}^{*p} = \varepsilon_{ij}^p - \varepsilon^p\delta_{ij}$$

where ε_{ij}^p are the components of the plastic strain rates, σ_{ij} are the stress tensor components, and e_{ij} are the strain tensor components. Summation over repeated subscripts is assumed throughout, where the Latin subscripts $i, j, \ldots$ take on the values 1 to 3, and the Greek letters $\alpha, \beta, \ldots$ the values 1, 2.

In the stress space the flow surface is a closed cone (Fig. 1) which expands depending on e_{ij}^p and ε_{ij}^p. The sides surface and bottom of the cone are described by the first and second equations in (1.1), respectively. The position of the flow surface for $\varepsilon_{ij}^p = 0$ is shown by the solid line in Fig. 1. The dashed line shows the instantaneous state of the flow surface for $\varepsilon_{ij}^p \neq 0$. The relationships (1.1) are a generalization of the model proposed in [3] for soils when the influence of viscosity and plastic strain on the change in the loading surface is taken into account. It is henceforth assumed that the strains are small and comprised of two parts, elastic and plastic, $e_{ij} = e_{ij}^e + e_{ij}^p$. The elastic strain tensor is related to the stresses by Hooke's law

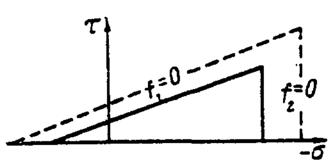

Fig. 1

$$\sigma_{ij} = \lambda e_{kk}^e \delta_{ij} + 2\mu e_{ij}^e$$

where λ, μ are the elastic parameters of the medium. The spatial coordinates are assumed to be Cartesian. The strains are expressed in terms of displacements by means of the Cauchy formula

$$e_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i})$$

The plastic strain rate tensor is connected with the stress tensor by the plasticity condition and the associated plastic flow law, which in the case of a state of stress corresponding to the side surface $f_1 = 0,\ f_2 < 0$ of the flow condition is

$$\varepsilon_{ij}^p = \psi_1 \frac{\partial f_1}{\partial \sigma_{ij}} = \frac{2\psi_1 s_{ij}}{1 + 2\eta_1\psi_1} + \frac{{}^4/_3\,\alpha\psi_1}{1 - {}^4/_3\alpha\eta_2\psi_1}\,(k_0 - \alpha\sigma + \beta\,(e^p)) \qquad (1.2)$$

Eliminating the parameter ψ_1 from the plasticity condition expressed by the first equation in (1.1), and the associated plastic flow law (1.2), we find the dependence of the plastic strain rate on the stress

$$\varepsilon_{ij}^p = \frac{s_{ij}}{\eta} - \frac{\sqrt{2}\,(k_0 - \alpha\sigma + \beta)\,s_{ij}}{\eta\sqrt{s_{kl}s_{kl}}} + \frac{\sqrt{2}\,\alpha\sqrt{s_{kl}s_{kl}}}{3\eta}\,\delta_{ij} - \frac{2\alpha\,(k_0 - \alpha\sigma + \beta)}{3\eta}\,\delta_{ij},$$

$$\eta = \eta_1 + \frac{2}{3}\,\alpha\eta_2 \qquad (1.3)$$

If the state of stress in the body corresponds to the bottom $f_2 = 0,\ f_1 < 0$ of the flow surface, then the associated flow law is

$$\varepsilon_{ij}^p = \psi_2 \frac{\partial f_2}{\partial \sigma_{ij}} = -\frac{1}{3}\,\psi_2\delta_{ij} \qquad (1.4)$$

Eliminating the parameter from the second equation of (1.1) and (1.4), we find

$$\varepsilon_{ij}^p = \frac{\sigma + \sigma_0 + \mu_1\,|\,e^p\,|}{\eta_3}\,\delta_{ij} \qquad (1.5)$$

For the angular points of the flow surface the plastic strain rate tensor ε_{ij}^p is related to the stress tensor by the plasticity condition (1.1) and the associated plastic flow law relationships

$$\varepsilon_{ij}^p = \psi_1 \frac{\partial f_1}{\partial \sigma_{ij}} + \psi_2 \frac{\partial f_2}{\partial \sigma_{ij}} \qquad (1.6)$$

Eliminating the parameters ψ_1 and ψ_2 from (1.6) and (1.1), we obtain

$$\varepsilon_{ij}^p = \frac{s_{ij}}{\eta_1} - \frac{\sqrt{2}\,s_{ij}\,\{(k_0 - \alpha\sigma + \beta)\,\eta_3 + (\sigma + \sigma_0 + \mu_1\,|\,e^p\,|)\,\eta_2\}}{\eta_3\eta_1\sqrt{s_{kl}s_{kl}}} + \frac{\sigma + \sigma_0 + \mu\,|\,e^p|}{\eta_3}\,\delta_{ij} \qquad (1.7)$$

The equations describing the dynamic behavior of the medium are

$$\sigma_{ij,\,j} - \rho v_{i,\,t} = 0 \qquad (1.8)$$

$$\sigma_{ij,t} = \lambda v_{k,k}\delta_{ij} + \mu\,(v_{i,j} + v_{j,i}) - 2\mu\varepsilon_{ij}^p - \lambda\varepsilon_{kk}^p\delta_{ij}$$

$$\varepsilon_{ij}^p = 0, \qquad \text{if} \quad \begin{cases} (s_{ij} - \eta_1\varepsilon^{*p})\,(s_{ij} - \eta_1\varepsilon_{ij}^{*p}) - 2\,(k_0 - \alpha\sigma + \beta + \eta_2\,|\,\varepsilon^p\,|)^2 < 0 \\ -\sigma - \sigma_0 - \mu_1\,|\,e^p\,| - \eta_3\,|\,\varepsilon^p\,| < 0 \end{cases}$$

$$\varepsilon_{ij}^p = \varphi\,(\sigma_{ij}), \qquad \text{if} \quad (s_{ij} - \eta_1\varepsilon_{ij}^{*p})\,(s_{ij} - \eta_1\varepsilon_{ij}^{*p}) - 2\,(k_0 - \alpha\sigma + \beta + \eta_2\,|\,\varepsilon^p\,|)^2 \geqslant 0$$

$$\text{or} \quad -\sigma - \sigma_0 - \mu_1\,|\,e^p\,| - n_3\,|\,\varepsilon^p\,| \geqslant 0$$

Here v_i are the components of the displacement rate vector, and $\varphi\,(\sigma_{ij})$ is the right side of (1.3), (1.5) or (1.7).

It can be shown [4] that in media described by the rheological equations (1.1) there exist two kinds of waves of strong discontinuity, irrotational and equivoluminal, which are propagated at the elastic wave velocities $c_1 = \sqrt{(\lambda + 2\mu) / \rho}$ and $c_2 = \sqrt{\mu / \rho}$. The following relationships

$$-c\,[\sigma_{ij}] = \lambda\,[v_k]\,\nu_k + \mu\,([v_i]\,\nu_j + [v_j]\,\nu_i), \qquad [\sigma_{ij}]\,\nu_j + \rho c\,[v_i] = 0 \quad (1.9)$$

are satisfied for discontinuities in the velocities and stresses on these waves. Here ν_i is the unit vector normal to the wave surface, where $[v_i] = \omega\nu_i$ on the irrotational waves, where $\omega = [v_i]\,\nu_i$, and $[v_i]\,\nu_i = 0$ on the equivoluminal waves.

2. Let us represent the solution in the plastic flow domain of the medium for the velocities, stresses and strains as series in powers of h, where h is the distance along the normal behind the front of the sutface of strong discontinuity

$$f = f^-|_\Sigma - hf^-_{,n}|_\Sigma + {}^1/_2\, f^-_{,nn}|_\Sigma - \ldots \quad (2.1)$$

Here $f^-|_\Sigma$, $f^-_{,n}|_\Sigma$, ... are values of the functions on the front of this wave. The series (2.1) for the velocity behind the front has the form

$$v_i = v_i^+|_\Sigma - [v_i] + h\,([v_{i,n}] - v^+_{i,n}|_\Sigma) + \ldots \quad (2.2)$$
$$[v_i] = (v_i^+ - v_i^-)|_\Sigma$$

Here the plus superscript refers to the value of the function ahead of the strong discontinuity wave front, and the minus superscript to the value of the function behind the wave front. Analogous series can be written for the stresses and plastic strains.

Let us write (1.8) in the discontinuities. From the first-order kinematic and geometric compatibility conditions [5] ($x_i = x_i\,(y^1, y^2, t)$ is the equation of the wave surface, $x_{i\beta} = \partial x_i / \partial y^\beta$, $g^{\alpha\beta}$ are components of the contravariant metric tensor of the wave surface and y^1, y^2 are curvilinear coordinates on the surface)

$$[f_{,i}] = [f_{,n}]\,\nu_i + g^{\alpha\beta}\,[f]_{,\alpha}x_{i\beta}, \qquad [f_{,t}] = -c\,[f_{,n}] + \frac{\delta}{\delta t}\,[f] \quad (2.3)$$

and the relations (2.2), we obtain an equation to determine the zero terms of the series (2.2) following [4]

$$\frac{\delta\omega}{\delta t} = c_1\Omega\omega + \frac{1}{2\rho c_1}\,(\lambda\,[\varepsilon^p_{kk}] + 2\mu\,[\varepsilon_{ij}{}^p]\,\nu_i\nu_j) \quad (2.4)$$
$$\frac{\delta\,[v_i]}{\delta t} = c_2\Omega\,[v_i] + c_2\,([\varepsilon_{ij}{}^p]\,\nu_j - [\varepsilon^p_{kl}]\,\nu_k\nu_l\nu_i)$$

Here

$$\Omega = (\Omega_0 - K_0ct)\,(1 - 2\Omega_0ct + K_0c^2t^2)^{-1}$$

Ω is the mean curvature of the wave surface at any instant, Ω_0, K_0 are the mean and Gaussian curvatures at the initial instant. To evaluate the remaining terms of the ray expansion (2.2), let us determine the discontinuities in the velocity derivatives of any order $[v^{(k)}_{i,n\ldots n}]$ on the surface of strong discontinuity. Members of the series of the plastic strains and stresses also depend on the velocity discontinuities and velocity derivatives on this surface.

To do this, let us differentiate (1.8) m times with respect to the normal to the wave surface ($f_{,n} = f_{,i}\nu_i$, where ν_i are components of the unit vector normal to the wave front). Taking their difference at different sides of the wave surface, we obtain

$$
\begin{gathered}
[\sigma_{ij,\,in\ldots n}^{(m+1)}] = \lambda\,[v_{k,\,kn\ldots n}^{(m+1)}] + \mu\,([v_{i,\,jn\ldots n}^{(m+1)}] + [v_{j,\,in\ldots n}^{(m+1)}]) - \\
\lambda\,[\varepsilon_{kk,\,n\ldots n}^{p\,(m)}]\,\delta_{ij} - 2\mu\,[\varepsilon_{ij,\,n\ldots n}^{p\,(m)}] \\
[\sigma_{ij,\,jn\ldots n}^{(m+1)}] - \rho\,[v_{i,\,in\ldots n}^{(m+1)}] = 0
\end{gathered}
\tag{2.5}
$$

Not only the velocities and stresses, but also their derivatives of any order undergo a discontinuity on the strong discontinuity wave. The compatibility conditions must be satisfied for the discontinuities of the derivatives of these functions on the wave. Generalizing the derivation of the first- and second-order compatibility conditions [5], we obtain the geometric and kinematic compatibility conditions of any order for the discontinuities of the derivatives $[f_{,in\ldots n}^{(k)}]$, $[f_{,in\ldots n}^{(k)}]$, which have the following form:

$$
[f_{,\,in\ldots n}^{(k)}] = [f_{,\,n\ldots n}^{(k)}]\,\nu_i + g^{\alpha\beta}x_{i\beta}\,[f_{,\,n\ldots n}^{(k-1)}]_{,\,\alpha} + (k-1)\,g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\,[f_{,\,ln\ldots n}^{(k-1)}]\,x_{l\tau}x_{i\beta}
\tag{2.6}
$$

$$
[f_{,\,tn\ldots n}^{(k)}] = -\,c\,[f_{,\,n\ldots n}^{(k)}] + \frac{\delta}{\delta t}\,[f_{,\,n\ldots n}^{(k-1)}]
\tag{2.7}
$$

where $b_{\alpha\sigma}$ are components of the second fundamental quadratic form of the surface. Substituting the kinematic compatibility condition (2.7) into (2.5), we obtain for the discontinuities in the quantities $[v_{i,tn\ldots n}^{(m+1)}]$ and $[\sigma_{ij,tn\ldots n}^{(m+1)}]$.

$$
\begin{gathered}
c\,[\sigma_{ij,\,n\ldots n}^{(m+1)}] = \frac{\delta}{\delta t}\,[\sigma_{ij,\,n\ldots n}^{(m)}] - \lambda\,[v_{k,\,kn\ldots n}^{(m+1)}]\,\delta_{ij} - \mu\,([v_{i,\,jn\ldots n}^{(m+1)}] + \\
[v_{j,\,in\ldots n}^{(m+1)}]) + \lambda\,[\varepsilon_{kk,\,n\ldots n}^{p\,(m)}]\,\delta_{ij} + 2\mu\,[\varepsilon_{ij,\,n\ldots n}^{p\,(m)}] \\
[\sigma_{ij,\,jn\ldots n}^{(m+1)}] - \rho\left(-c\,[v_{i,\,n\ldots n}^{(m+1)}] + \frac{\delta}{\delta t}\,[v_{i,\,n\ldots n}^{(m)}]\right) = 0
\end{gathered}
\tag{2.8}
$$

Using the geometric compatibility conditions (2.6) for the discontinuities of the quantities $[v_{j,in\ldots n}^{(m+1)}]$ and $[\sigma_{ij,jn\ldots n}^{(m+1)}]$, and eliminating quantities $[\sigma_{ij,\,n\ldots n}^{(m+1)}]$, $[\sigma_{ij,\,n\ldots n}^{(m)}]$, from (2.8), we obtain

$$
\begin{gathered}
(\lambda+\mu)\,[v_{k,\,n\ldots n}^{(m+1)}]\,\nu_i\nu_k + (\mu-\rho c^2)\,[v_{i,\,n\ldots n}^{(m+1)}] = -\,2\rho c\,\frac{\delta}{\delta t}\,[v_{i,\,n\ldots n}^{(m)}] - \\
\lambda g^{\alpha\beta}x_{k\beta}\,[v_{k,\,n\ldots n}^{(m)}]_{,\,\alpha}\nu_i - \mu g^{\alpha\beta}x_{i\beta}[v_{k,\,n\ldots n}^{(m)}]_{,\,\alpha}\nu_k - \lambda g^{\alpha\beta}x_{i\beta}\,[v_{k,\,kn\ldots n}^{(m)}]_{,\,\alpha} - \\
\mu g^{\alpha\beta}x_{j\beta}\,[v_{i,\,jn\ldots n}^{(m)}]_{,\,\alpha} - \mu g^{\alpha\beta}x_{j\beta}\,[v_{j,\,in\ldots n}^{(m)}]_{,\,\alpha} - \frac{\delta}{\delta t}\,\{g^{\alpha\beta}x_{j\beta}\,[\sigma_{ij,\,n\ldots n}^{(m-1)}]_{,\,\alpha} + \\
(m-1)\,g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{j\beta}\,[\sigma_{ij,\,kn\ldots n}^{(m-1)}]\,x_{k\tau}\} + \rho\,\frac{\delta^2}{\delta t^2}\,[v_{i,\,n\ldots n}^{(m-1)}] - \\
\lambda m g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{k\beta}\,[v_{k,\,ln\ldots n}^{(m)}]\,x_{l\tau}\nu_i + g^{\alpha\beta}x_{j\beta}\left(\frac{\delta}{\delta t}\,[\sigma_{ij,\,n\ldots n}^{(m-1)}]\right)_{,\,\alpha} - \\
\mu m g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{i\beta}\,[v_{k,\,ln\ldots n}^{(m)}]\,x_{l\tau}\nu_k + c m g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{j\beta}\,[\sigma_{ij,\,ln\ldots n}^{(m)}]\,x_{l\tau} + \\
g^{\alpha\beta}x_{j\beta}\,(\lambda\,[\varepsilon_{kk,\,n\ldots n}^{p\,(m-1)}]\,\delta_{ij} + 2\mu\,[\varepsilon_{ij,\,n\ldots n}^{p\,(m-1)}])_{,\,\alpha} + \lambda\,[\varepsilon_{kk,\,n\ldots n}^{p\,(m)}]\,\nu_i + 2\mu\,[\varepsilon_{ij,\,n\ldots n}^{p\,(m)}]\,\nu_j, \\
m = 1, 2, 3, \ldots
\end{gathered}
\tag{2.9}
$$

To determine $[\varepsilon_{ij}^{p}]$, $[\varepsilon_{ij,n}^{p}]$, ..., $[\varepsilon_{ij,n\ldots n}^{p(m)}]$ it is necessary to use the relationships (1.3), (1.5) or (1.7). Multiplying (2.9) by ν_i and summing over i, then setting $\rho c^2 = \lambda + 2\mu$, we obtain a differential equation for the change in the quantities $[v_{i,n\ldots n}^{(m)}]\,\nu_i = \omega_n^{(m)}$ on the irrotational wave

$$
\begin{gathered}
2\rho c \frac{\delta \omega_n^{(m)}}{\delta t} - 2\rho c^2 \Omega \omega_n^{(m)} + (\lambda + \mu) g^{\alpha\beta} ([v_{k,n\ldots n}^{(m)}] x_{k\alpha})_{,\beta} - \\
\rho \frac{\delta^2 \omega_n^{(m-1)}}{\delta t^2} + \mu g^{\alpha\beta} x_{j\beta} \nu_i \{ g^{\sigma\tau} x_{j\tau} [v_{i,n\ldots n}^{(m-1)}]_{,\sigma} + (m-1) g^{pq} g^{\sigma\tau} b_{p\sigma} x_{jq} [v_{i,ln\ldots n}^{(m-1)}] x_{l\tau} + \\
g^{\sigma\tau} x_{i\tau} [v_{j,n\ldots n}^{(m-1)}]_{,\sigma} + (m-1) g^{pq} g^{\sigma\tau} b_{p\sigma} x_{iq} [v_{j,ln\ldots n}^{(m-1)}] x_{l\tau} \}_{,\alpha} + \\
\frac{\delta}{\delta t} \{ g^{\alpha\beta} x_{j\beta} [\sigma_{ij,n\ldots n}^{(m-1)}] + (m-1) g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j\beta} [\sigma_{ij,kn\ldots n}^{(m-1)}] x_{k\tau} \} \nu_i - \\
g^{\alpha\beta} x_{j\beta} \nu_i \left(\frac{\delta}{\delta t} [\sigma_{ij,n\ldots n}^{(m-1)}] \right)_{,\alpha} + \lambda m g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{k\beta} [v_{k,n\ldots n}^{(m-1)}]_{,\tau} + \qquad (2.10) \\
\lambda m (m-1) g^{\alpha\beta} g^{\sigma\tau} g^{pq} b_{\alpha p} b_{q\tau} x_{k\beta} [v_{k,in\ldots n}^{(m-1)}] x_{i\sigma} - cm(m-1) g^{\alpha\beta} g^{\sigma\tau} g^{pq} b_{\tau q} b_{\alpha\sigma} x_{j\beta} \nu_i \times \\
[\sigma_{ij,ln\ldots n}^{(m-1)}] x_{lp} - cm g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j\beta} [\sigma_{ij,n\ldots n}^{(m-1)}]_{,\tau} \nu_i - 2\mu g^{\alpha\beta} x_{j\beta} \nu_i [\varepsilon_{ij,n\ldots n}^{p\,(m-1)}]_{,\alpha} - \\
\lambda [\varepsilon_{kk,n\ldots n}^{p\,(m)}] - 2\mu [\varepsilon_{ij,n\ldots n}^{p\,(m)}] \nu_i \nu_j = 0, \qquad m = 1, 2, 3, \ldots
\end{gathered}
$$

Let us set $m-1$ instead of m and $\rho c^2 = \lambda + 2\mu$ in the system (2.9). Multiplying the equations obtained by $x_{i\varkappa}$ and summing over i, we find the relationships to determine the components of the vector $[v_{i,n\ldots n}^{(m)}]$ on the irrotational wave which are tangential to the front

$$
\begin{gathered}
(\mu - \rho c^2) [v_{i,n\ldots n}^{(m)}] x_{i\varkappa} = -\mu \omega_{n,\varkappa}^{(m-1)} - \mu g^{\alpha\beta} b_{\beta\varkappa} [v_{i,n\ldots n}^{(m-1)}] x_{i\alpha} + \\
x_{i\varkappa} \nu_j \frac{\delta}{\delta t} [\sigma_{ij,n\ldots n}^{(m-1)}] + c(m-1) g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j\beta} x_{i\varkappa} [\sigma_{ij,ln\ldots n}^{(m-1)}] x_{l\tau} - \qquad (2.11) \\
\mu (m-1) g^{\sigma\tau} b_{\varkappa\sigma} [v_{j,ln\ldots n}^{(m-1)}] x_{l\tau} \nu_j - \rho c x_{i\varkappa} \frac{\delta}{\delta t} [v_{i,n\ldots n}^{(m-1)}] - c (2\Omega x_{i\varkappa} \nu_j + \\
g^{\alpha\beta} b_{\varkappa\alpha} x_{j\beta} \nu_i) [\sigma_{ij,n\ldots n}^{(m-1)}] - 2\mu [\varepsilon_{ij,n\ldots n}^{p\,(m-1)}] \nu_i x_{i\varkappa} + c g^{\alpha\beta} (x_{j\beta} x_{i\varkappa} [\sigma_{ij,n\ldots n}^{(m-1)}])_{,\alpha}
\end{gathered}
$$

Assuming $\rho c^2 = \mu$, in the system (2.9), we have, after multiplication by ν_i and summation over i

$$
\begin{gathered}
(\lambda + \mu) [v_{k,n\ldots n}^{(m+1)}] \nu_k = \\
-2\rho c \frac{\delta \omega_n^{(m)}}{\delta t} - \lambda g^{\alpha\beta} x_{k\beta} [v_{k,n\ldots n}^{(m)}]_{,\alpha} - \mu g^{\alpha\beta} x_{j\beta} \nu_i [v_{i,jn\ldots n}^{(m)}]_{,\alpha} - \\
\mu g^{\alpha\beta} x_{j\beta} [v_{j,in\ldots n}^{(m)}]_{,\alpha} \nu_i - \frac{\delta}{\delta t} \{ g^{\alpha\beta} x_{j\beta} \nu_i [\sigma_{ij,n\ldots n}^{(m-1)}]_{,\alpha} + (m-1) g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j\beta} \nu_i \times \\
[\sigma_{ij,kn\ldots n}^{(m-1)}] x_{k\tau} \} + \rho \frac{\delta^2}{\delta t^2} [v_{i,n\ldots n}^{(m-1)}] \nu_i - \lambda m g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{k\beta} [v_{k,ln\ldots n}^{(m)}] x_{l\tau} + \\
c(m-1) m g^{\alpha\beta} g^{\sigma\tau} g^{pq} b_{\alpha\sigma} b_{\tau q} x_{j\beta} [\sigma_{ij,ln\ldots n}^{(m-1)}] x_{lp} \nu_i + 2\mu g^{\alpha\beta} x_{j\beta} \nu_i [\varepsilon_{ij,n\ldots n}^{p\,(m-1)}]_{,\alpha} + \\
\lambda [\varepsilon_{kk,n\ldots n}^{p\,(m)}] + 2\mu [\varepsilon_{ij,n\ldots n}^{p\,(m)}] \nu_i \nu_j + g^{\alpha\beta} x_{j\beta} \left(\frac{\delta}{\delta t} [\sigma_{ij,n\ldots n}^{(m-1)}] \right)_{,\alpha} \nu_i + \qquad (2.12) \\
cm g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{j\beta} [\sigma_{ij,n\ldots n}^{(m-1)}]_{,\tau} \nu_i
\end{gathered}
$$

$$
\omega^{(m)} = [v_{i,n\ldots n}^{(m)}] \nu_i, \qquad \omega_\alpha^{(m)} = [v_{i,n\ldots n}^{(m)}] x_{i\alpha}
$$

Here $\omega_n^{(m)}$ is the normal, and $\omega_\alpha^{(m)}$ the tangential components of the vector $[v_{i,n\ldots n}^{(m)}]$. Then, eliminating $[v_{k,n\ldots n}^{(m+1)}] \nu_k$ from the system (2.9) for $\rho c^2 = \mu$, we obtain a system of differential equations to determine the tangential components of the vector $[v_{i,n\ldots n}^{(m)}]$ on the equivoluminal wave

$$
\begin{gathered}
2\rho c \frac{\delta}{\delta t}(g^{\alpha\beta}\omega_{\alpha}^{(m)}x_{i\beta}) - 2\mu\Omega\,(g^{\alpha\beta}\omega_{\alpha}^{(m)}x_{i\beta}) + (\lambda+\mu)\,g^{\alpha\beta}x_{i\beta}\omega_{n,\alpha}^{(m)} + \\
\lambda g^{pq}\{g^{\alpha\beta}x_{k\beta}([v_{k,n\ldots n}^{(m-1)}]_{,\alpha} + (m-1)\,g^{\sigma\tau}b_{\alpha\sigma}\,[v_{k,ln\ldots n}^{(m-1)}]\,x_{l\tau})\}_{,p}x_{iq} + \\
2\mu g^{pq}\{g^{\alpha\beta}x_{j\beta}([v_{i,n\ldots n}^{(m-1)}]_{,\alpha} + (m-1)\,g^{\sigma\tau}b_{\alpha\sigma}\,[v_{i,ln\ldots n}^{(m-1)}]\,x_{l\tau})\}_{,p}x_{jq} - \\
2\mu g^{pq}\{g^{\alpha\beta}x_{j\beta}([v_{k,n\ldots n}^{(m-1)}]_{,\alpha} + (m-1)\,g^{\sigma\tau}b_{\alpha\sigma}\,[v_{k,ln\ldots n}^{(m-1)}]\,x_{l\tau})\}_{,p}x_{jq}\nu_k\nu_i + \\
2\mu g^{pq}\{g^{\alpha\beta}x_{i\beta}([v_{j,n\ldots n}^{(m-1)}]_{,\alpha} + (m-1)\,g^{\sigma\tau}b_{\alpha\sigma}\,[v_{j,ln\ldots n}^{(m-1)}]\,x_{l\tau})\}_{,p}x_{jq} - \\
2\mu g^{pq}\{g^{\alpha\beta}x_{k\beta}([v_{j,n\ldots n}^{(m-1)}]_{,\alpha} + (m-1)\,g^{\sigma\tau}b_{\alpha\sigma}\,[v_{j,ln\ldots n}^{(m-1)}]\,x_{l\tau})\}_{,p}x_{jq}\nu_k\nu_i - \\
\rho\frac{\delta^2}{\delta t^2}(g^{\alpha\beta}\omega_{\alpha}^{(m-1)}x_{i\beta}) + \mu m g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{i\beta}\nu_k([v_{k,n\ldots n}^{(m-1)}]_{,\tau} + \qquad (2.13)\\
(m-1)\,g^{pq}\,b_{\tau p}\,[v_{k,ln\ldots n}^{(m-1)}]\,x_{lq}) - g^{\alpha\beta}x_{j\beta}\left(\frac{\delta}{\delta t}[\sigma_{ij,n\ldots n}^{(m-1)}]\right)_{,\alpha} + \\
cm\,(m-1)\,g^{\alpha\beta}g^{\sigma\tau}g^{pq}b_{\alpha\sigma}b_{\tau q}x_{j\beta}\,[\sigma_{kj,ln\ldots n}^{(m-1)}]\,x_{lp}\nu_k\nu_i - cmg^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{j\beta}\,[\sigma_{ij,n\ldots n}^{(m-1)}]_{,\tau} + \\
cmg^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{j\beta}\,[\sigma_{kj,n\ldots n}^{(m-1)}]_{,\tau}\nu_k\nu_i - \lambda g^{\alpha\beta}x_{i\beta}\,[\varepsilon_{kk,n\ldots n}^{p\,(m-1)}]_{,\alpha} - 2\mu g^{\alpha\beta}x_{j\beta}([\varepsilon_{ij,n\ldots n}^{p\,(m-1)}]_{,\alpha} - \\
[\varepsilon_{kj,n\ldots n}^{p\,(m-1)}]_{,\alpha}\nu_k\nu_i) + g^{\alpha\beta}x_{j\beta}\left(\frac{\delta}{\delta t}[\sigma_{kj,n\ldots n}^{(m-1)}]\right)_{,\alpha}\nu_k\nu_i - 2\mu\,([\varepsilon_{ij,n\ldots n}^{p\,(m)}]\,\nu_j - \\
[\varepsilon_{kj,n\ldots n}^{p\,(m)}]\,\nu_k\nu_j\nu_i) - cm\,(m-1)\,g^{\alpha\beta}g^{\sigma\tau}g^{pq}b_{\alpha\sigma}b_{\tau q}x_{j\beta}\,[\sigma_{ij,ln\ldots n}^{(m-1)}]\,x_{lp} = 0
\end{gathered}
$$

The relationship (2.12), where $m-1$ must be taken instead of m, yields the magnitude of the component, normal to the equivoluminal wave front, of the vector $[v_{i,n\ldots n}^{(m)}]$.

Thus, (2.11), (2.12) are differential relationships from which the tangential (in the irrotational wave case) and the normal (in the equivoluminal case) components of the vector $[v_{i,n\ldots n}^{(m)}]$ are determined by means of the known $[v_{i,n\ldots n}^{(m-1)}]$ and $[\sigma_{ij,n\ldots n}^{(m-1)}]$. By virtue of (2.11) and (2.12), the right sides of (2.10), (2.13) can be considered known quantities if only $[v_i]$, $[v_{i,n}]$, $\ldots$, $[v_{i,n\ldots n}^{(m-1)}]$ and $[\sigma_{ij}]$, $[\sigma_{ij,n}^{(1)}]$, $\ldots$, $[\sigma_{ij,n\ldots n}^{(m-1)}]$ are determined. The zero order terms of $[v_i]$ and, according to (1.10), of $[\sigma_{ij}]$ are determined from (2.4) for the irrotational wave and from (2.3) for the equivoluminal wave. It is necessary to use the relationships (1.3), (1.5) or (1.7) to determine $[\varepsilon_{ij}^{p}]$, $[\varepsilon_{ij,n}^{p(1)}]$, $\ldots$, $[\varepsilon_{ij,n\ldots n}^{p(m)}]$ in terms of the discontinuities in the stresses and their derivatives.

Thus, members of the ray expansions for the velocities, stresses, and plastic strains, and therefore, the values of these functions behind the shock wave are determined successively from (2.10) – (2.13), (2.4), (2.8) and (1.10).

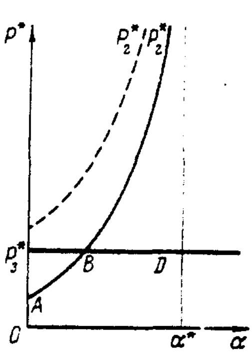

Fig. 2

3. As an illustration, let us consider the propagation of a spherical loading wave in an unstressed space. Let the stress $\sigma_{rr} = -P(t)$, $\sigma_{r\varphi} = 0$, $\sigma_{r\theta} = 0$, $P(t) > 0$ be given on the boundary $r = r_0$. Depending on the values of the parameters defining the medium σ_0 / k_0, η_1/η_3, η_2/η_3, ν and the pressure $P(0)/k_0$, the material behind the front of a longitudinal wave can be strained elastically or plastically at $t = 0$, corresponding to different parts of the flow surface, the side surface, bottom, or edge. For pressures $P(0) < P_0$, where P_0 is determined by the inequalities

$$
f_1^{\circ} = s_{ij}s_{ij} - 2\,(k_0 - \alpha\sigma)^2 < 0, \qquad f_2^{\circ} = -\sigma - \sigma_0 < 0 \qquad (3.1)
$$

the strain state behind the wave front is elastic. The graphs $P^* = P_3^*(\alpha, \nu)$ and $P^* = P_2^*(\alpha, \nu)$, corresponding to the solution of the equations $f_1^\circ = 0$, $f_2^\circ = 0$, $P^* = P(0)/k_0$ are constructed in Fig. 2. As is shown, the material will be in the elastic state behind the wave at the initial instant when the pressure P^* is in the domain bounded by the curve ABD if $k_0 / \sigma_0 < \alpha^*$, or below the line $P^* = P_3^*$ if $k_0 / \sigma_0 \geqslant \alpha^*$. Here

$$P_3^* = 3\frac{\sigma_0}{k_0}\left(\frac{1-\nu}{1+\nu}\right), \qquad P_2^* = 3\left(\frac{1-\nu}{1+\nu}\right)\frac{1}{-\alpha+\alpha^*}, \qquad \alpha^* = \sqrt{3}\,\frac{1-2\nu}{1+\nu}$$

The heavy solid lines correspond to the values $k_0 / \sigma_0 < \alpha^*$ and the dashed to the values $k_0 / \sigma_0 \geqslant \alpha^*$. The stressed state behind the wave corresponding to the side surface of the flow condition is determined by the pressure $P(0)$ satisfying the relations

$$f_1 = (s_{ij} - \eta_1\varepsilon_{ij}^{*p})(s_{ij} - \eta_1\varepsilon_{ij}^{*p}) - 2(k_0 - \alpha\sigma + \eta_2|\varepsilon^p|)^2 = 0$$
$$f_2 = -\sigma - \sigma_0 - \eta_3|\varepsilon^p| < 0 \tag{3.2}$$

The plastic state corresponding to the bottom of the flow surface is determined by the pressure $P(0)$ from the relationships

$$f_2 = 0, \qquad f_1 < 0 \tag{3.3}$$

The state of stress determined by the initial pressure $P(0)$ which satisfies the following relations

$$f_1 = 0, \quad f_2 \geqslant 0 \quad \text{or} \quad f_2 = 0, \quad f_1 \geqslant 0$$

corresponds to the edge of the flow surface.

Presented in Figs. 3 and 4 are the curves

$$P^* = P_3^*(\alpha, \nu), \quad P^* = P_2^*(\alpha, \nu), \quad P_1^* - 3\left(\frac{1-\nu}{1+\nu}\right)\left(\frac{\sigma_0}{k_0} - 1\right)\frac{1}{\alpha + 1 - \alpha^*}$$

for $\sigma_0 / k_0 > 1$, $\sqrt{3}(1 - 2\nu) / (1 + \nu) \geqslant 1$ and $\sigma_0 / k_0 < 1$, $\sqrt{3}(1 - 2\nu)/(1 + \nu) > k_0 / \sigma_0$, respectively; they separate the domain of the elastic state from the plastic state corresponding to different parts of the flow surface.

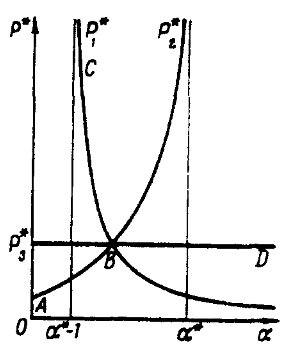

Fig. 3

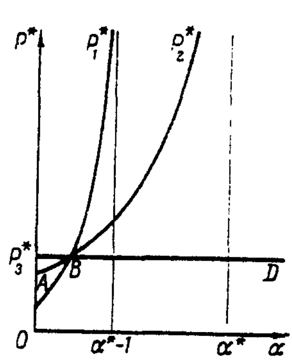

Fig. 4

A combined analysis of the inequalities presented shows that the plastic state of strain behind the wave corresponds to the side surface of the flow condition if the initial pressure is in the domain bounded by the curve ABC and the P^* axis (Figs. 3 and 4). A plastic state satisfying the bottom of the flow surface corresponds to pressures in the domain bounded by the curve CBD. The initial pressures on the curve $P_1^*(\alpha, \nu)$

(the section CB) cause a plastic flow corresponding to the edge of the flow surface. If the initial pressures P^* are below the curve ABD, then the behavior of the material behind the wave is elastic.

Now, let us determine the velocity, stress, and strain behind the surface of the strong discontinuity being propagated by considering the initial pressure to be such that a plastic strain corresponding to the side surface of the flow condition $f_1 = 0, f_2 < 0$ for the case $\beta(e^p) = 0, \eta_2(e^p) = \text{const}$ is realized behind the wave. The dependences between the plastic strain rates and the stresses become

$$\varepsilon_{ij}^p = \frac{s_{ij}}{\eta} - \frac{\sqrt{2}\, s_{ij}}{\eta}\frac{(k_0 - \alpha\sigma)}{\sqrt{s_{kl}s_{kl}}} + \frac{\sqrt{2}\,\alpha\sqrt{s_{kl}s_{kl}}}{3\eta}\delta_{ij} - \frac{2\alpha(k_0 - \alpha\sigma)}{3\eta}\delta_{ij}$$

Let us seek the velocity behind the front as

$$v_r(r,\ t) = -\omega + \omega_n^{(1)}(r_0 + c_1 t - r) - {}^1/_2\,\omega_n^{(2)}(r_0 + c_1 t - r)^2 + \ldots \qquad (3.4)$$

Stress and strain series are formed analogously, but their members depend on discontinuities in the velocity and its derivatives on the surface of strong discontinuity. Hence, to determine the stresses and strains in the plastic flow domain of the medium it is sufficient to find terms of the ray expansion (3.4).

Writing down the jump in the plastic strain rates $[\varepsilon_{ij}^p]$ and substituting them into (2.4) for the zero term of the series, we have after transformations

$$\frac{\delta\omega}{\delta t} + \left\{\frac{c_1'}{r_0 + c_1 t} + \varkappa_1\right\}\omega = -\frac{k_0 D}{3\rho c_1\eta} \qquad (3.5)$$

$$D = 2\sqrt{3}\,\mu - \alpha(3\lambda + 2\mu), \qquad \varkappa_1 = D^2 / 9\rho c_1\eta, \qquad \eta = \eta_1 + {}^2/_3\alpha\eta_2$$

Here r_0 is the radius of the wave surface at the instant $t = 0$, $r = r_0 + c_1 t$ is the equation of the wave surface at an arbitrary instant. Integrating (3.5), we obtain

$$\omega = (r_0 + c_1 t)^{-1}\{Ae^{-\varkappa_1 t} - k_0 D(r_0 + c_1 t - c_1/\varkappa_1)(3\rho c_1\eta\varkappa_1)^{-1}\}$$

$$A = \omega^\circ r_0 + k_0 D(r_0 - c_1/\varkappa_1)(3\rho c_1\eta\varkappa_1)^{-1}$$

Here ω° is the value of $[v_i]v_i$ at the initial instant. Evaluating the quantities $[\varepsilon_{ij}^p]$ and substituting them into (2.10) where $m = 1$, we obtain an equation to determine the first member of the series (3.4)

$$\frac{\delta\omega_n^{(1)}}{\delta t} + \left\{\frac{c_1}{r_0 + c_1 t} + \varkappa_1\right\}\omega_n^{(1)} = F_1(t)$$

$$F_1(t) = \{3\varkappa_1^2 / 2c_1 - 2\varkappa_1 E / 3\rho c_1\eta + 2c_1(r_0 + c_1 t)^{-2}\}\,\omega +$$

$$\varkappa_1 k_0 D / 2\rho c_1^2\eta + 2k_0 DE / 9\rho c_1^2\eta^2 + k_0 D\{6\rho c_1\eta(r_0 + c_1 t)^{-1} - 2\sqrt{3}\,\mu\, k_0/\rho c\eta(r_0 + ct)$$

$$E = 3\mu + \alpha^2(3\lambda + 2\mu)$$

Its solution is

$$\omega^{(1)} = (r_0 + c_1 t)^{-1} e^{-\varkappa_1 t}\left\{\omega_{n0}^{(1)} r_0 + \int_0 F_1(t)(r_0 + c_1 t)e^{\varkappa_1 t}\,dt\right\}$$

The tangential components of the vector $[v_{i,n}]$ are determined from the system (2.12) for $m = 1$ and equal zero. Behind the wave let the plastic strain corresponding to the bottom of the flow surface $f_2 = 0$, $f_1 < 0$ be realized. The plastic strain rate in the plastic flow zone of the medium is calculated by means of (1.5). The damping equation for the zero member of the series (3.4), in this case, has the form

$$\frac{\delta\omega}{\delta t} + \left\{\frac{c_1}{r_0 + c_1 t} + \varkappa_2\right\}\omega = -\frac{(3\lambda + 2\mu)\sigma_0}{2\rho c_1 \eta_3}$$

Its solution is written as

$$\omega = (r_0 + c_1 t)^{-1}\{A_2 e^{-\varkappa_2 t} - 3\sigma_0 c_1 (r_0 + c_1 t - c_1/\varkappa_2)(3\lambda + 2\mu)^{-1}\}$$

$$A_2 = \omega^\circ r_0 + 3\sigma_0 c_1 (r_0 - c/\varkappa_2)(3\lambda + 2\mu)^{-1}, \qquad \varkappa_2 = (3\lambda + 2\mu)^2/(6\rho c^2 \eta_3)$$

The first member of the series (3.4) is determined from the equation

$$\frac{\delta\omega_n^{(1)}}{\delta t} + \left\{\frac{c_1}{r_0 + c_1 t} + \varkappa_2\right\}\omega_n^{(1)} = F_2(t)$$

$$\omega_n^{(1)} = (r_0 + c_1 t)^{-1} e^{-\varkappa_2 t}\left\{r_0 \omega_{n0}^{(1)} + \int_0^t F_2(t)(r_0 + c_1 t) e^{\varkappa_2 t}\, dt\right\}$$

$$F_2(t) = \left\{\frac{2c_1}{(r_0 + c_1 t)^2} + \frac{3\varkappa_2^2}{2c_1} - \frac{3\lambda + 2\mu}{c_1 \eta_3}\right\}\omega + \frac{3(3\lambda + 2\mu)\sigma_0\varkappa_2}{4\rho c_1^2 \eta_3} + \frac{(3\lambda + 2\mu)\sigma_0}{4\rho c_1 \eta_3 (r_0 + c_1 t)} - \frac{3\delta_0 \varkappa_2}{\eta_3}$$

When the initial pressure $P(0)$ is such that plastic strain corresponding to the edge of the flow surface $f_1 = 0$, $f_2 = 0$, is realized behind the wave, the equation to determine the zero term of the series (3.4) is

$$\frac{\delta\omega}{\delta t} + \left\{\frac{c_1}{r_0 + c_1 t} + \varkappa_3\right\}\omega = -\frac{(3\lambda + 2\mu)\sigma_0}{2\rho c_1 \eta_3} - \frac{2\mu(k_0\eta_3 + \sigma_0\eta_2)}{\sqrt{3}\,\rho c_1 \eta_3 \eta_1} \tag{3.6}$$

$$\omega = (r_0 + c_1 t)^{-1}\{A e^{-\varkappa_3 t} + B(r_0 + c_1 t - c_1/\varkappa_3)\varkappa_3^{-1}\}$$

$$A_3 = \omega^\circ r_0 + B(r_0 \varkappa_3 - c_1)$$

$$\varkappa_3 = \frac{1}{2\rho c_1}\left\{\frac{(3\lambda + 2\mu)^2}{3\rho c_1 \eta_3} + \frac{8\mu^2}{3\rho c_1 \eta_1} - \frac{4\mu(3\lambda + 2\mu)(\alpha\eta_3 - \eta_2)}{3\sqrt{3}\,\rho c_1 \eta_3 \eta_1}\right\}$$

Here B is the right side of (3.6). The first member of the series is determined from the following equation in this case:

$$\frac{\delta\omega_n^{(1)}}{\delta t} + \left\{\frac{c_1}{r_0 + c_1 t} + \varkappa_3\right\}\omega_n = F_3(t)$$

$$F_3(t) = \omega\left\{\frac{c_1}{(r_0 + c_1 t)^2} + \frac{3\varkappa_3^2}{2c_1} - \frac{8\mu^2}{6\rho c_1^3 \eta_1^2} + \frac{4\mu^2(3\lambda + 2\mu)(\alpha\eta_3 - \eta_2)}{6\sqrt{3}\,c_1^3 \rho \eta_1^2 \eta_3} - \right.$$
$$\left. \frac{(3\lambda + 2\mu)^3}{6\rho c_1^3 \eta_3^2} + \frac{2\mu(\alpha\eta_3 - \eta_2)(3\lambda + 2\mu)^2}{6\sqrt{3}\,\rho c_1^3 \eta_3^2 \eta_1} + \frac{2\mu(3\lambda + 2\mu)(\alpha\eta_3 - \eta_2)}{2\sqrt{3}\,\rho c_1^2 \eta_3 \eta_1 (r_0 + c_1 t)}\right\} -$$
$$\frac{2\sqrt{3}\,\mu(k_0\eta_3 + \sigma_0\eta_2)}{\rho c_1 \eta_3 \eta_1 (r_0 + c_1 t)} + \frac{3\varkappa_3(3\lambda + 2\mu)\sigma_0}{4\rho c_1^3 \eta_3} - \frac{4\mu^2(k_0\eta_3 + \sigma_0\eta_2)}{4\rho c_1 (r_0 + c_1 t)\eta_3} -$$
$$\frac{(3\lambda + 2\mu)\sigma_0}{2\rho c_1^2 \eta_3}\left(\frac{3\lambda + 2\mu}{\eta_3} - \frac{4\mu(\alpha\eta_3 - \eta_2)}{\sqrt{3}\,\eta_3 \eta_1}\right) + \frac{\sqrt{3}\,\mu\varkappa_3(k_0\eta_3 + \sigma_0\eta_2)}{\rho c_1^2 \eta_3 \eta_1}$$

$$\omega_n^{(1)} = (r_0 + c_1 t)^{-1} \exp(-\varkappa_3 t) \left\{ r_0 \omega_{n0}^{(1)} + \int_0^t F_3(t) \exp(\varkappa_3 t)(r_0 + c_1 t)\, dt \right\}$$

The tangential components of the vector $[v_{i,n}]$, are missing in both this and preceding case.

Limiting ourselves to two members, we can write down the solution for the velocity (and therefore, for the stress and strain) in the plastic flow domain of the medium to the accuracy of small terms of the order of $(r_0 + ct - r)^2$.

Let us determine the initial values ω°, $\omega_{n0}^{(1)}$, in the solution for $\omega(t)$ and $\omega_n^{(1)}(t)$. The normal component of the velocity v_r given by the series (3.4) on a sphere of radius r_0 should equal $P(t) / \rho c_1$, i.e.

$$P(t) / \rho c_1 = -\omega + \omega_n^{(1)} (c_1 t) - {}^1/_2\, \omega_n^{(2)} (ct)^2 + \ldots \qquad (3.7)$$

Setting $t = 0$ in (3.7), we obtain

$$\omega(0) = \omega^\circ = P(0) / \rho c_1$$

Differentiating (3.7) with respect to t and setting $t = 0$, we have

$$\omega_n^{(1)}(0) = \omega_{n0}^\circ = P'(0) / \rho c_1^2 + \omega'(0) / c_1$$

Here ω' is determined from the solution for $\omega(t)$. Proceeding in this manner, we find all the initial values needed to determine the members of the series (3.4).

BIBLIOGRAPHY

1. Alekseev, A.S., Babich, V.M. and Gel'chinskii, B.Ia., Ray method of calculating wave front intensity. In: Questions of the Dynamic Theory of Seismic Wave Propagation. Leningrad Univ. Press, 1961.
2. Achenbach, J.D. and Reddy, D.P., Note on wave propagation in linearly viscoelastic media. Z. angew. Math. und Phys., Bd.18, 1967.
3. Jenike, A.W. and Shield, R.T., On the plastic flow of Coulomb's solids beyond original failure. J. Appl. Mech., Vol.26, №4, 1959.
4. Bykovtsev, G.I. and Verveiko, N.D., On wave propagation in elastic viscoplastic media. Inzh. Zh., Mekh. Twerd. Tela, №4, 1966.
5. Thomas, T., Plastic Flow and Fracture in Solids. Moscow, "Mir", 1961.

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